(3) we have

$$u = r\mu^{-1}\mu^{\cdot}, \quad p = \alpha\mu^{\gamma}\mu_z z^{\alpha-1} + c\mu^{\gamma}$$
$$\rho = \alpha(\alpha-1)\mu z^{\alpha-3}, \quad j = (\gamma-1)^{-1}\mu^{\gamma-1} z^{\alpha}\mu_z'$$
$$z = \mu r, \quad k = \alpha r^{-1}.$$

The suggestion to look for exact solutions of (1) with a linear dependence of the velocity on the spatial coordinate was made by V.P. Korobeinikov.

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# THE INFLUENCE TENSOR FOR AN ELASTIC MEDIUM WITH POISSON'S RATIO VARYING IN ONE DIRECTION*

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We construct an analogue of the known Kelvin-Somigliana tensor for an unbounded elastic medium with a Poisson's ratio that varies in one direction and a constant shear modulus. We deduce the corresponding force tensor. We also consider the effect of the temperature. The effect of inhomogeneity is demonstrated by examples.

**1. Initial relations.** We can attach the following form to the resolving equations of the linear theory of elasticity of the inhomogeneous medium under consideration in a Cartesian coordinate system $x_i$ $(i = 1, 2, 3)$ /2/:

$$\Delta m = -\Phi_{1,1} - \Phi_{2,2} + \Phi_{3,3}, \quad \Delta n = -\Phi_{1,2} + \Phi_{2,1} \tag{1.1}$$

$$\Delta k = (1-\nu)^{-1}[m_{,33} + \nu(\Phi_{1,1} + \Phi_{2,2}) + (1+\nu)\alpha\theta] - \Phi_{3,3} \tag{1.2}$$
$$(X_i = 2\mu\gamma^2\Phi_i \ (i = 1, 2), \ X_3 = 2\mu\Phi_{3,33}).$$

Here $m, k, n$ are resolving potential functions, $X_i$ are components of the volume force vector, $\Phi_i$ are arbitrary volume force potential functions, $\theta$ is the temperature, $\alpha$ is the coefficient of linear expansion, $\nu$ is Poisson's ratio, $\mu$ is the shear modulus, $\Delta$ is the Laplace operator and $\gamma^2$ is the two-dimensional Laplace operator (in the variables $x_1$ and $x_2$). Partial derivatives are indicated by a comma followed by the index of the corresponding variable.

The components of the dislocation vector $u_i$ and the stress tensor $\sigma_{ij}$ are expressed in terms of the potential functions according to the formulae

$$u_i = (k+m)_{,i} + 2(\varepsilon_{ip3}n_{,p} - \delta_{3i}m_{,3}) \tag{1.3}$$
$$\sigma_{ij} = 2\mu\{(k+m)_{,ij} + \varepsilon_{jp3}n_{,ip} + \varepsilon_{ip3}n_{,pj} - \delta_{3i}m_{,3j} - \delta_{3j}m_{,3i} +$$
$$(1-2\nu)^{-1}[\nu\Delta(k+m) - 2\nu m_{,33} - (1+\nu)\alpha\theta]\}$$

($\delta_{ij}$ is the Kronecker delta and $\varepsilon_{ijp}$ are components of the Levi-Civita tensor).

In the relations we have noted, Poisson's ratio is everywhere taken to be an arbitrarily differentiable or, in the general case, partially-differentiable function of one variable $x_3$,

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and the shear modulus is taken to be a constant quantity. With a constant Poisson's ratio and no volume forces and temperature in relations (1.1)-(1.3), it is not difficult to recognize a special case of the Papkovich–Neuber representation /1/, and a well-known representation /3/ in the theory of homogeneous elastic media.

The following representation will prove useful in the future:

$$[1-\nu(x_3)]^{-1} = D + \beta(x_3) \tag{1.4}$$

where the constant $D$ is chosen so that the integral

$$C = \lim_{h\to\infty} \int_{-h}^{h} \beta(\xi)\, d\xi \tag{1.5}$$

is well-defined.

If, for instance,

$$\nu(x_3) = \nu_1 e(-x_3) + \nu_2 e(x_3) \tag{1.6}$$

($e(t)$ is a Heaviside function /4/, $\nu_i$ are fixed constants), then

$$D = \frac{1}{2}\left(\frac{1}{1-\nu_1} + \frac{1}{1-\nu_2}\right),\quad H = \frac{1}{2}\left(\frac{1}{1-\nu_2} - \frac{1}{1-\nu_1}\right),\quad \beta = H \operatorname{sign} x_3,\quad C = 0.$$

**2. Auxiliary singular solutions.** Let $\mathbf{x}(x_i)$, $\mathbf{y}(y_i)$ be, respectively, the point of observation and the source point (the singular action point), $\delta(x_i - y_i)$ the density of the singular action (the Dirac function /4/) and let

$$r^2 = \sum_{i=1}^{2} (x_i - y_i)^2,\quad R^2 = \sum_{i=1}^{3} (x_i - y_i)^2,\quad \omega = (x_3 - y_3) \ln \frac{x_3 - y_3 + R}{r} - R.$$

The following transcendental-operator decompositions and equalities hold:

$$\omega = -\cos\gamma(x_3 - y_3)\, r,\quad R^{-1} = \cos\gamma(x_3 - y_3)\, r^{-1} \tag{2.1}$$

$$\omega_{,33} = -\gamma^2\omega = R^{-1},\quad \Delta\omega = 0,\quad \Delta\omega_{,33} = -4\pi\delta \tag{2.2}$$

(the latter holds in a generalized function space /4/).

Suppose it is required to find a solution of the non-uniform Laplace equation

$$\Delta f = (D + \beta)\nabla R^{-1}\cdot\mathbf{l} \tag{2.3}$$

($\mathbf{l}(l_i)$ is an arbitrarily-oriented unit vector) that obeys the symmetry condition and the boundary condition

$$f(\mathbf{x}, \mathbf{y}) = f(\mathbf{y}, \mathbf{x}),\quad \lim_{|\mathbf{x}|\to\infty} \nabla f = 0. \tag{2.4}$$

Taking account of (2.1), we transform Eq.(2.3) into an ordinary differential equation

$$f_{,33} + \gamma^2 f = (D + \beta)\nabla \cos\gamma(x_3 - y_3)\, r^{-1}\cdot\mathbf{l}$$

($\gamma$ is treated as though it was a constant parameter), whose solution is known /5/. The following transition from a transcendental solution to one expressed explicitly in terms of elementary functions is also accomplished on the basis of (2.1). We add some solutions of the uniform Laplace equation to the solution so obtained in order to satisfy conditions (2.4). As a result, the required solution takes the form

$$f = {}^1/_2\mathbf{L}\cdot\mathbf{l};\quad \mathbf{L} = \nabla\Omega - \mathbf{n}_3\Omega' \tag{2.5}$$

$$\Omega = DR + C\ln r + \int_{y_3}^{x_3} \beta\, d\xi\, \omega_{,3} + \frac{1}{2}\sum_{i=1}^{4}\int_{h_i}^{t_i} \beta\omega_{\xi,3}\, d\xi$$

$$\Omega' = \sum_{i=1}^{4}\int_{h_i}^{t_i} \beta(\xi) R_\xi^{-1}\, d\xi$$

$$h_1 = h_3 = -\infty,\quad h_2 = h_4 = \infty,\quad t_1 = t_2 = x_3,\quad t_3 = t_4 = y_3.$$

The addition of the index $\xi$ to a function here and henceforth indicates the replacement of the argument $x_3 - y_3$ of this function by the new argument $x_3 + y_3 - 2\xi$.

**3. Action of the unit force.** We assume that a unit force with density acts on the elastic space. Taking account of (2.2), we can write

$$4\pi\mathbf{X} = -\Delta\gamma^2\omega\mathbf{l} = \Delta\omega_{,33}\mathbf{l}$$

from which follows

$$\Phi_1/l_1 = \Phi_2/l_2 = -\Phi_3/l_3 = q\Delta\omega,\quad q = (8\pi\mu)^{-1}. \tag{3.1}$$

We obtain from (1.1)

$$m = -q\nabla\omega\cdot\mathbf{l},\quad n = -q\nabla\times\mathbf{n}_3\omega\cdot\mathbf{l}. \tag{3.2}$$

From Eq.(1.2), taking account of the dependence (3.1), (3.2) and of (2.5) we obtain

$$k = -{}^1/_2 q\,[\mathbf{L}\cdot\mathbf{l} + 2\,(\nabla\cdot\mathbf{l}\omega - 2\mathbf{n}_3\cdot\nabla\mathbf{l}\omega\cdot\mathbf{n}_3)]. \tag{3.3}$$

The components of the influence tensor and of the force tensor are determined by the relations

$$\mathbf{u} = \mathbf{U}\cdot\mathbf{l},\quad \sigma = \mathbf{F}\cdot\mathbf{l}.$$

From (1.3) and the solutions (3.2), (3.3) it follows that

$$\mathbf{U} = 2q\,(\mathbf{E}R^{-1} - {}^1/_4\nabla\mathbf{L}) \tag{3.4}$$
$$\mathbf{F} = -(4\pi)^{-1}\{{}^1/_2\,\nabla\nabla\mathbf{L} - [\nu(1-\nu)^{-1}\mathbf{E}\nabla + \nabla\mathbf{E} + \mathbf{n}_i\nabla\mathbf{n}_i]\,R^{-1}\}$$

($E$ is a unit tensor). We can immediately verify the correctness of the reciprocity principle and the balance condition

$$\mathbf{U}(\mathbf{x},\mathbf{y}) = \mathbf{U}^T(\mathbf{y},\mathbf{x}),\quad \nabla\cdot\mathbf{F} = -\mathbf{E}\delta.$$

For a homogeneous space, $(L_i = DR_{,i})$, representation (3.4) reduces to some well-known ones /1, 4/.

**4. Effect of temperature.** It is appropriate to set $\theta = \delta$ in (1.2). Then, in accordance with properties (2.2)

$$k_\theta = -\frac{A}{4\pi}\frac{1}{R},\quad A = \frac{1+\nu(y_3)}{1-\nu(y_3)}\,\alpha. \tag{4.1}$$

The components of the temperature displacement vector and of the stress tensor take the following form, in accordance with (1.3) and (4.1):

$$\mathbf{u}_\theta = -\frac{A}{4\pi}\nabla R^{-1},\quad \sigma_\theta = -\frac{\mu A}{2\pi}(\nabla\nabla - \mathbf{E}\Delta)\,R^{-1}. \tag{4.2}$$

It follows from the dependence we have obtained that the field of displacements and stresses in an elastic space with a variable Poisson's ratio depends only on the last singular thermal action at the point.

Suppose we are given a temperature distribution $\theta(\mathbf{x})$ in the volume $V$, and let the temperature be zero outside this volume. As in a homogenous space, we introduce the potential

$$\chi = (4\pi)^{-1}\iiint\limits_V R^{-1}A\theta(\mathbf{y})\,dv_y.$$

In accordance with (4.2)

$$\mathbf{u}_\theta(\mathbf{x}) = -\nabla\chi,\quad \sigma_\theta(\mathbf{x}) = -2\mu\,(\nabla\nabla - \mathbf{E}\Delta)\,\chi. \tag{4.3}$$

We can also arrive at the first expression in (4.3) by a different route, based on the following formula which follows from a theorem in /1/ concerning reciprocity

$$\mathbf{u}_\theta(\mathbf{x}) = 2\mu\alpha\iiint\limits_V [1+\nu(y_3)]\,[1-2\nu(y_3)]^{-1}\theta(\mathbf{y})\,\nabla_y\cdot\mathbf{U}(\mathbf{y},\mathbf{x})\,dv_y$$

and the following easily-checked relations.

$$\Delta\Omega = \frac{2}{1-\nu}\frac{1}{R},\quad \Delta\Omega' = 2\beta_{,3}\frac{1}{R},\quad \nabla\cdot\mathbf{U} = q\,\frac{1-2\nu}{1-\nu}\nabla\frac{1}{R}.$$

**5. Examples.** Let the space be characterized by a stepwise-varying Poisson's ratio of the form (1.6). In the given case, the functions $\Omega, \Omega'$ in (2.5) will be

$$\Omega = DR + H\,[(|x_3| - |y_3|)\,\omega_{,3} + (x_3 + y_3)\ln r] - {}^1/_2 H\,(\omega - \omega_0)(\operatorname{sign} x_3 + \operatorname{sign} y_3)$$
$$\Omega' = H\,[(\omega_0 + \omega)_{,3}\operatorname{sign} x_3 + (\omega_0 - \omega)_{,3}\operatorname{sign} y_3 + 2\ln r]$$
$$\omega_0 = \omega_\xi|_{\xi=0}.$$

In accordance with (3.4), we deduce expressions for the components of the influence tensor and the force tensor

$$U_{ij} = 2q\,\{R^{-1}\delta_{ij} - {}^1/_4 DR_{,ij} - {}^1/_4 H\,[(|x_3| - |y_3|)\,\omega_{,3ij} + (x_3+y_3)\ln r]_{,ij} - 2(\ln r)_{,i}\delta_{3j} + [{}^1/_2(\omega_0-\omega)_{,j} + (\omega-\omega_0)_{,3}\delta_{3j}]_{,i}(\operatorname{sign} x_3 + \operatorname{sign} y_3) + (\omega_{,j}\delta_{3i} - \omega_{,i}\delta_{3j})_{,3}\operatorname{sign} x_3]\}$$
$$F_{ijk} = -(4\pi)^{-1}\{{}^1/_2 DR_{,ijk} + (1-D)(1/R)_{,k}\delta_{ij} - (1/R)_{,i}\delta_{jk} - \delta_{ig}\delta_{gk}(1/R)_{,j} + {}^1/_2 H\,[(|x_3| - |y_3|)\,\omega_{,3ijk} + [(x_3+y_3)\ln r]_{,ijk} - 2\delta_{3k}(\ln r)_{,ij} + [{}^1/_2(\omega_0-\omega)_{,k} + (\omega-\omega_0)_{,3}\delta_{3k}]_{,ij}(\operatorname{sign} x_3 + \operatorname{sign} y_3) + [-2(1/R)_{,k}\delta_{ij} + (\omega_{,kj}\delta_{3i} - \omega_{,ij}\delta_{3k} + \omega_{,ik}\delta_{3j})_{,3}]\operatorname{sign} x_3]\}.$$

In particular, at the origin of coordinates $(x_i = 0)$ under the assumption $y_1 = y_2 = 0$, we have

$$F_{333} = \frac{1}{8\pi\,|y_3|^2}\,[H + 2(1 + D)\,\mathrm{sign}\,y_3]$$

and under the assumption $y_2 = y_3 = 0$ we have

$$F_{331} = -\frac{1}{8\pi\,|y_1|^2}\,(2 - D)\,\mathrm{sign}\,y_1 .$$

The difference in the values of the stresses calculated according to the expressions given here between the cases with $\nu_1 = 0, \nu_2 = 0.45$ (for an inhomogeneous medium) and $\nu_1 = \nu_2 = 0.225$ (for a homogeneous medium) reaches 14-17%.

Now let the state of stress in the inhomogeneous space under consideration be created by the heating of two non-interesting spherical volumes $V_1$ and $V_2$ with respective radii $a_1$ and $a_2$ and centres $(0, 0, a_1)$ and $(0, 0, -a_2)$ to the constant respective temperatures $\Theta_1$ and $\Theta_2$. Inside each volume, the material is homogeneous and the states of stress and of deformation of the medium are, as has earlier been established, indifferent to the behaviour of Poisson's ratio outside the given heated volume, and so by analogy with well-known expressions for a homogeneous medium /1/ we can write

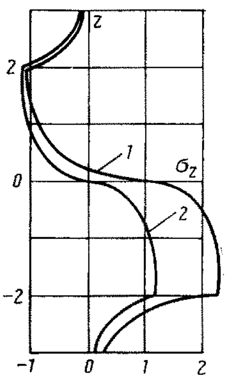


Fig.1

$$\chi_j = \begin{cases} {}^1/_2 A_j \Theta_j (a_j^2 - {}^1/_3 R_j^2), & \mathbf{x} \in V_j \\ {}^1/_3 A_j \Theta_j a_j^3 / R_j, & \mathbf{x} \,\overline{\in}\, V_j \end{cases}$$

$$A_j = \frac{1 + \nu_j}{1 - \nu_j}\,\alpha, \quad R_j = \{x_1^2 + x_2^2 + [x_3 - (-1)^{j-1} a_j]^2\}^{1/2}$$

$$j = 1, 2 .$$

Fig.1 shows the distribution of the relative normal stress $\sigma_z = 3\sigma_{33\theta}/(4\mu\alpha\Theta)$ along the axis $z = x_3/a$ that passes through the centres of the spherical volumes, with $\Theta_1 = -\Theta_2 = \Theta = \mathrm{const}, a_1 = a_2 = a$ for an inhomogeneous medium ($\nu_1 = 0.1, \nu_2 = 0.4$, curve *1*) and for a homogeneous medium ($\nu_1 = \nu_2 = 0.1$, curve *2*). We can see the important influence of the inhomogeneity of Poisson's ratio on the distribution of thermal stresses.

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